Monday: Turing machines

We are ready to introduce a formal model that will capture a notion of general purpose computation.

- *Similar to DFA, NFA, PDA*: input will be an arbitrary string over a fixed alphabet.
- *Different from NFA, PDA*: machine is deterministic.
- *Different from DFA, NFA, PDA*: read-write head can move both to the left and to the right, and can extend to the right past the original input.
- *Similar to DFA, NFA, PDA*: transition function drives computation one step at a time by moving within a finite set of states, always starting at designated start state.
- *Different from DFA, NFA, PDA*: the special states for rejecting and accepting take effect immediately.

(See more details: Sipser p. 166)

Formally: a Turing machine is $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where δ is the **transition function**

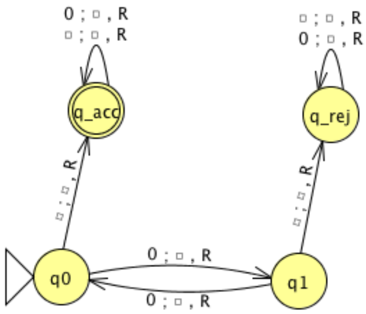
$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$$

The **computation** of M on a string w over Σ is:

- Read/write head starts at leftmost position on tape.
- Input string is written on $|w|$-many leftmost cells of tape, rest of the tape cells have the blank symbol. **Tape alphabet** is Γ with $\llcorner\!\lrcorner \in \Gamma$ and $\Sigma \subseteq \Gamma$. The blank symbol $\llcorner\!\lrcorner \notin \Sigma$.
- Given current state of machine and current symbol being read at the tape head, the machine transitions to next state, writes a symbol to the current position of the tape head (overwriting existing symbol), and moves the tape head L or R (if possible).
- Computation ends **if and when** machine enters either the accept or the reject state. This is called **halting**. Note: $q_{accept} \neq q_{reject}$.

The **language recognized by the Turing machine** M, is $L(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$, which is defined as

$$\{w \in \Sigma^* \mid \text{computation of } M \text{ on } w \text{ halts after entering the accept state}\}$$



Formal definition:

Sample computation:

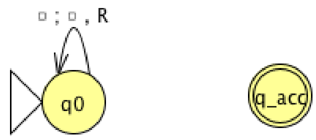
$q0 \downarrow$						
0	0	0	␣	␣	␣	␣

The language recognized by this machine is . . .

Describing Turing machines (Sipser p. 185) To define a Turing machine, we could give a

- **Formal definition**: the 7-tuple of parameters including set of states, input alphabet, tape alphabet, transition function, start state, accept state, and reject state; or,
- **Implementation-level definition**: English prose that describes the Turing machine head movements relative to contents of tape, and conditions for accepting / rejecting based on those contents.
- **High-level description**: description of algorithm (precise sequence of instructions), without implementation details of machine. As part of this description, can "call" and run another TM as a subroutine.

Fix $\Sigma = \{0, 1\}$, $\Gamma = \{0, 1, \text{␣}\}$ for the Turing machines with the following state diagrams:



Example of string accepted:
Example of string rejected:

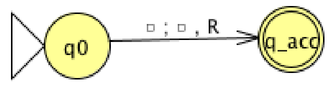
Implementation-level description

High-level description

Example of string accepted:
Example of string rejected:

Implementation-level description

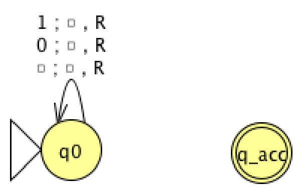
High-level description



Example of string accepted:
Example of string rejected:

Implementation-level description

High-level description



Example of string accepted:
Example of string rejected:

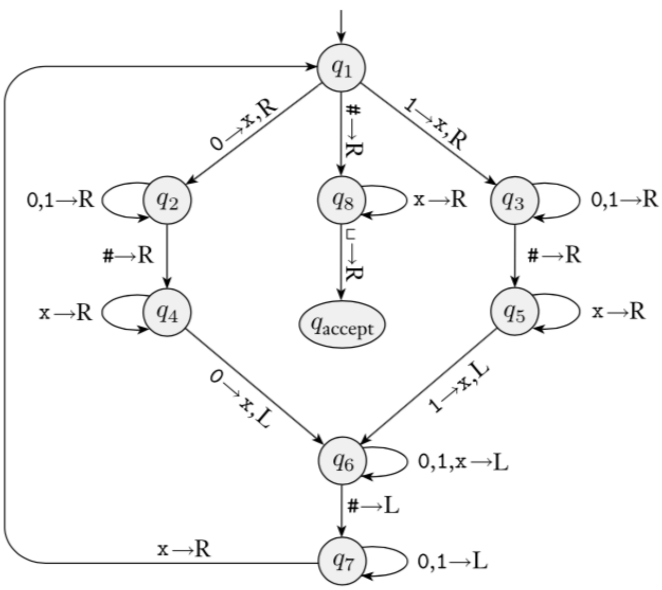
Implementation-level description

High-level description

Wednesday: Describing Turing machines and algorithms

Sipser Figure 3.10

Conventions in state diagram of TM: $b \to R$ label means $b \to b, R$ and all arrows missing from diagram represent transitions with output $(q_{reject}, ␣, R)$



Computation on input string 01#01

$q_1 \downarrow$						
0	1	#	0	1	␣	␣

Implementation level description of this machine:

> Zig-zag across tape to corresponding positions on either side of # to check whether the characters in these positions agree. If they do not, or if there is no #, reject. If they do, cross them off.
>
> Once all symbols to the left of the # are crossed off, check for any un-crossed-off symbols to the right of #; if there are any, reject; if there aren't, accept.

The language recognized by this machine is

$$\{w\#w \mid w \in \{0,1\}^*\}$$

High-level description of this machine is

Recall: High-level descriptions of Turing machine algorithms are written as indented text within quotation marks. Stages of the algorithm are typically numbered consecutively. The first line specifies the input to the machine, which must be a string.

Extra practice

Computation on input string 01#1

$q_1 \downarrow$						
0	1	#	1	␣	␣	␣

A language L is **recognized by** a Turing machine M means

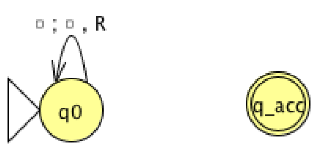
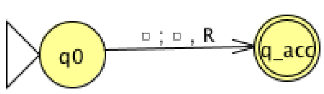
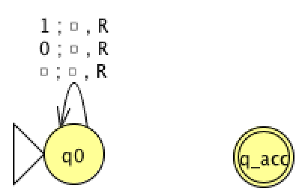
A Turing machine M **recognizes** a language L means

A Turing machine M is a **decider** means

A language L is **decided by** a Turing machine M means

A Turing machine M **decides** a language L means

Fix $\Sigma = \{0, 1\}$, $\Gamma = \{0, 1, \text{␣}\}$ for the Turing machines with the following state diagrams:

 Decider? Yes / No	 Decider? Yes / No
 Decider? Yes / No	 Decider? Yes / No

Friday: Decidable and Recognizable Languages

A **Turing-recognizable** language is a set of strings that is the language recognized by some Turing machine. We also say that such languages are recognizable.

A **Turing-decidable** language is a set of strings that is the language recognized by some decider. We also say that such languages are decidable.

An **unrecognizable** language is a language that is not Turing-recognizable.

An **undecidable** language is a language that is not Turing-decidable.

True or **False**: Any decidable language is also recognizable.

True or **False**: Any recognizable language is also decidable.

True or **False**: Any undecidable language is also unrecognizable.

True or **False**: Any unrecognizable language is also undecidable.

True or **False**: The class of Turing-decidable languages is closed under complementation.

Using formal definition:

Using high-level description:

Church-Turing Thesis (Sipser p. 183): The informal notion of algorithm is formalized completely and correctly by the formal definition of a Turing machine. In other words: all reasonably expressive models of computation are equally expressive with the standard Turing machine.

Definition: A language L over an alphabet Σ is called **co-recognizable** if its complement, defined as $\Sigma^* \setminus L = \{x \in \Sigma^* \mid x \notin L\}$, is Turing-recognizable.

Theorem (Sipser Theorem 4.22): A language is Turing-decidable if and only if both it and its complement are Turing-recognizable.

Proof, first direction: Suppose language L is Turing-decidable. WTS that both it and its complement are Turing-recognizable.

Proof, second direction: Suppose language L is Turing-recognizable, and so is its complement. WTS that L is Turing-decidable.

Notation: The complement of a set X is denoted with a superscript c, X^c, or an overline, $\overline{X}$.

Claim: If two languages (over a fixed alphabet Σ) are Turing-decidable, then their union is as well.

Proof:

Claim: If two languages (over a fixed alphabet Σ) are Turing-recognizable, then their union is as well.

Proof:

Week 6 at a glance

Textbook reading: Chapter 3, Section 4.1

For Monday: Page 165-166 Introduction to Section 3.1

For Wednesday: Example 3.9 on page 173

For Friday: Page 184-185 Terminology for describing Turing machines

Make sure you can:

- Use and design automata both formally and informally, including DFA, NFA, PDA, TM.
  - Use precise notation to formally define the state diagram of DFA, NFA, PDA, TM.
  - Use clear English to describe computations of DFA, NFA, PDA, TM informally
  - Determine whether a language is recognizable by a (D or N) FA and/or a PDA
  - Motivate the definition of a Turing machine
  - Trace the computation of a Turing machine on given input
  - Describe the language recognized by a Turing machine
  - Determine if a Turing machine is a decider
  - Given an implementation-level description of a Turing machine
  - Use high-level descriptions to define and trace Turing machines
  - Apply dovetailing in high-level definitions of machines
  - State and use the Church-Turing thesis
- Classify the computational complexity of a set of strings by determining whether it is regular, context-free, decidable, or recognizable.
- Give examples of sets that are regular, context-free, decidable, or recognizable.

TODO:

Review quizzes based on class material each day.

Homework assignment 3 due this Thursday.

Project due next Thursday.