Agenda: Review session for finals

(i) NFA to DFA conversion

NFA (N):

```
 q1
|__\  a
  |   b
  |   e
 q2 ------- q3
 a, b
```

Equivalence DFA (D):

```
(0, 2, 8, 9, 9)
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Start state of N: q1,
Start state of D: E(\{q1\}) = \{q1, q2\}

defined in Thm 1.39 (pg 56) Sipser
Accept state(s) of $N$: \{q_1\}

Accept state(s) of $D$: \{\{q_1\}, \{q_1, q_2\}, \{q_2, q_3\}, \{q_1, q_2, q_3\}\}

\text{Sample computation of transition function:}

$d'(\{q_1\}, a) = \emptyset$ because there is no transition defined for $(q_1, a)$ in $N$.

$d'(\{q_3\}, a) = \{q_2\}$

$E(\{q_3\}) = \{q_1, q_2\}$ ✓
(2) Pumping Lemma for Regular Languages

A is a regular language

\[ \exists p \text{ (pumping length) such that } \]

\[ \forall s \in A, \text{ if } |s| \geq p, \text{ then } \]

\[ \exists x, y, z \text{ such that } \]

\[ S = xyz \]

\[ |y| > 0 \]

\[ |xy| \leq p \]

\[ \forall i \geq 0, \ xy^iz \in A \]

Can be used to prove non-regularity.

i.e \( A \) is a regular language \( \Rightarrow \exists p \) s.t all strings \( s \in A \) with \( |s| \geq p \) can be pumped.

\( \forall p \exists s \in A, |s| > p, s \) cannot be pumped \( \Rightarrow \) \( A \) is NOT regular.

Example

Show that \( L = \text{REP}\{0^n1^n | n \geq 1\} \) is not regular.

- Let \( p \) be an arbitrary positive integer, \( w \), i.e. \( p \)

  is not the pumping length for \( L \).

- Let \( s = 2^p1^p2 \)
- We have:

  a) $S \in L$ because between every pair of successive 2s in $S$ is a string in
     \[ \{0^n1^n \mid n > 1\} \]

  b) $|S| = 2p+2 > p$ (since $p$ is a positive integer)

- Consider strings $x, y, z$ such that $S = xyz$, $|y| > 0$, $|xy| \leq p$

  Case (1) $x = \varepsilon$, $y = 2$, $z = 0^p1^2$

  \[ xy yz = 220^p1^2 \notin L \]

  between these 2s is the string $\varepsilon \notin \{0^n1^n \mid n > 1\}$

  (2) $x = \varepsilon$, $y = 20^m$, $z = 0^{p-m}1^2$ ($0 < m < p$)

  \[ xy yz = 20^m2^m0^{p-m}1^2 = 20^m20^p1^2 \notin L \]

  between these 2s is the string $0^m \notin \{0^n1^n \mid n > 1\}$

  (3) $x = 20^k$, $y = 0^m$, $z = 0^{p-m-k}1^2$ ($k > 0$, $0 < m < p$)

  \[ xy yz = 20^k0^m0^m0^{p-m-k}1^2 \]

  \[ = 20^{p+m}1^2 \notin L \] (similar reason)

  \[ \therefore \text{For any } p, \text{ we have some counter example } S \text{ that cannot be pumped with pumping length } p. \]

  \[ L \text{ is non-regular.} \]
For each regular language $L$, the language
$$\{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are DFAs} \land L(M_1) \subseteq L \land L(M_2) \subseteq \overline{L} \}$$
is decidable. True / False?

Sol: True.

- Use set identity $X \subseteq Y \iff X \cup Y = Y$
- Since $L$ is regular, there is a DFA (say $A$) such that $L(A) = L$. Also, since regular languages are closed under complement, there is a DFA (say $B$) such that $L(B) = \overline{L}$.
- Since $\text{EQ}_{DFA}$ is decidable, there is a $\text{TM} \ (\text{say } M_{\text{EQ}})$ that decides $\text{EQ}_{DFA}$.

Define $\text{TM } S =$

"on input $W$

1. If $W$ is not a valid encoding $\langle M_1, M_2 \rangle$ of 2 DFAs then reject.
2. Construct DFA $D_1$; $L(D_1) = L(A) \cup L(M_1)$
3. Run $M_{\text{EQ}}$ on input $\langle D_1, A \rangle$. If it rejects, reject.
4. Construct DFA $D_2$; $L(D_2) = L(B) \cup L(M_2)$
5. Run $M_{\text{EQ}}$ on input $\langle D_2, B \rangle$. If it rejects, reject.
6. Accept.

(Cite sample solutions for full justification)
(4) Mapping reduction theorems / strategies & sample question.

If \( A \leq_m B \) then

(a) \( B \) is decidable \( \rightarrow \) \( A \) is decidable
(b) \( A \) is undecidable \( \rightarrow \) \( B \) is undecidable
(c) \( B \) is recognizable \( \rightarrow \) \( A \) is recognizable
(d) \( A \) is not recognizable \( \rightarrow \) \( B \) is not recognizable

We know:

- (a) \( A_{\text{TM}} \) is recognizable
- (b) \( \overline{A_{\text{TM}}} \) is not recognizable \( \implies \) \( A_{\text{TM}} \) is undecidable

To prove some language \( B \) is NOT recognizable:

- Show \( \overline{A_{\text{TM}}} \leq_m B \) (or) \( A_{\text{TM}} \leq_m \overline{B} \)

To prove some language \( B \) is NOT co-recognizable (i.e. \( \overline{B} \) is not recognizable)

- Show \( \overline{A_{\text{TM}}} \leq_m \overline{B} \) i.e. \( A_{\text{TM}} \leq_m B \)

To prove some language \( B \) is recognizable you can

- a) mapping reduce \( B \) to some known recognizable language
- or

- b) Construct a TM that recognizes \( B \).

(and similarly for proving \( B \) is decidable).
e.g. \( E_{\text{TM}} \) is not recognizable (Thm 5.30 Sipser)

Let's \( A_{\text{TM}} \leq_m \overline{E_{\text{TM}}} \)

\( F = " \text{on input } \langle m, w \rangle \text{ where } M \text{ is a TM, } w \text{ is a string} \)

1. Construct \( T \text{m } M_1 = " \text{on input } x, \text{ reject}" \).
2. Construct \( T \text{m } M_2 = " \text{on input } x \)
   1. Run \( M \) on input \( w \).
   2. If \( M \) accepts, accept.
3. Output \( \langle M_1, M_2 \rangle \)

If \( M \) accepts \( w \), \( M_2 \) accepts all strings so \( L(M_1) \neq L(M_2) \)
If \( M \) doesn't accept \( w \), \( M_2 \) does not accept any string, so \( L(M_1) = L(M_2) \)

(To show \( E_{\text{TM}} \) is not co-recognizable, change \( M_1 \) to "on any input, accept".)

5. Proof that \( \text{HALT}_{\text{TM}} \) is undecidable (pg 217 Sipser)

Suppose that \( \text{HALT}_{\text{TM}} \) is decidable, i.e. some \( T \text{m } H \) that decides it.

Construct \( T \text{m } S \) that decides \( A_{\text{TM}} \) as follows:

\( S = " \text{on input } \langle m, w \rangle \text{ where } M \text{ is a TM, } w \text{ is a string} \)

1. Run \( H \) on \( \langle m, w \rangle \). If \( H \) rejects, reject.
2. Simulate \( M \) on \( w \): If \( M \) accepts, accept.

If \( M \) rejects, reject.

This is a contradiction, since we know \( A_{TM} \) is not decidable.