CSE 105 Discussion – Week 4

PUMPING LEMMA AND PDA
Do non-regular languages exist?

Yes! Why?

Q1: What does it mean for a language to be regular? (might have multiple right answers)
   A. Finite
   B. Can be recognized by an NFA/DFA
   C. Can be described by a Regex

Q2: What is the cardinality of the set of all Regex over some alphabet?
   A. Finite
   B. Countable
   C. Uncountable
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Now, the set of all languages is uncountable since it is the powerset of an infinite set.
Also note that the set of languages NFA/DFA/Regex can describe are the same!
Intuitions about non-regular langs

- $0^*1^*$
- $\{1^n \ 0^m \mid n, m > 1\}$
- $\{0, \ 00, \ 0000, \ 00000000, \ \ldots\}$
- $\{0, \ 000, \ 00000, \ \ldots\}$
- $\{0^n \ 1^m \mid n > m > 0\}$
Intuitions about non-regular langs

- $0^*1^*$ regular
- $\{1^n 0^m \mid n, m > 1\}$ regular
- $\{0, 00, 0000, 00000000, \ldots\}$ non-regular
- $\{0, 000, 00000, \ldots\}$ regular
- $\{0^n 1^m \mid n > m > 0\}$ non-regular
Pumping Lemma

If $A$ is a regular language then there is a number $p$ (the pumping length) where if $s$ is any string in $A$ of length at least $p$ then $s$ may be divided into three pieces, $s = xyz$ such that

- $|y| > 0$, and  // The loop is nonempty
- For each $i \geq 0, xy^iz \in A$  // Pumping the loop any # of times creates other strings in $A$
- $|xy| \leq p$  // The loop appears in the first $p$ characters

For regular languages we can set $p > \#$ of states in DFA recognizing language
For finite languages we can set $p >$ length of longest string in language
Pumping Lemma T/F

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Are the following statements True or False?

- $L$ is regular $\rightarrow$ $L$ has a pumping length
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- \(|xy| \leq p\) \( // \) The loop appears in the first \( p \) characters

Are the following statements True or False?

- \( L \) is regular \( \rightarrow \) \( L \) has a pumping length \( \text{True} \), this is what the Pumping Lemma tells us

- \( L \) is not regular \( \rightarrow \) \( L \) does not have a pumping length
Pumping Lemma T/F

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Are the following statements True or False?

- $L$ is regular $\rightarrow$ $L$ has a pumping length  **True**, this is what the Pumping Lemma tells us
- $L$ is not regular $\rightarrow$ $L$ does not have a pumping length  **False**, this is the converse of Pumping Lemma
- $L$ has a pumping length $\rightarrow$ $L$ is regular
Pumping Lemma T/F

If \( A \) is a regular language then there is a number \( p \) (the pumping length) where if \( s \) is any string in \( A \) of length at least \( p \) then \( s \) may be divided into three pieces, \( s = xyz \) such that

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- \( L \) is not regular \( \rightarrow \) \( L \) does not have a pumping length  **False**, this is the converse of Pumping Lemma
- \( L \) has a pumping length \( \rightarrow \) \( L \) is regular  **False**, this is the inverse of Pumping Lemma
- \( L \) does not have a pumping length \( \rightarrow \) \( L \) is not regular
Pumping Lemma T/F

If \( A \) is a regular language then there is a number \( p \) (the pumping length) where if \( s \) is any string in \( A \) of length at least \( p \) then \( s \) may be divided into three pieces, \( s = xyz \) such that

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Are the following statements True or False?

- \( L \) is regular \( \rightarrow \) \( L \) has a pumping length \quad \text{True, this is what the Pumping Lemma tells us}
- \( L \) is not regular \( \rightarrow \) \( L \) does not have a pumping length \quad \text{False, this is the converse of Pumping Lemma}
- \( L \) has a pumping length \( \rightarrow \) \( L \) is regular \quad \text{False, this is the inverse of Pumping Lemma}
- \( L \) does not have a pumping length \( \rightarrow \) \( L \) is not regular \quad \text{True, this is the contrapositive of the Pumping Lemma}
  - This is the statement we use to prove a language is not regular
Proof Sketch

- Suppose a language is regular, then it must have a DFA that recognizes it.
- DFA has finite amount of states, let’s say k.
- Let s be a string of length $n \geq k$.
- Suppose s is accepted, that means after $n$ transitions, we land in an accept state.
- Though the journey to accept state, we’ve visited $n+1$ states including the start.
- Now, $n+1 > k$, so at least one state has been visited twice.
- Let’s say the we visited $q_1, q_2 q_i, \ldots q_i, \ldots$ with $q_i$ visited at least two times.
- This shows there is a cycle. We can revisit the cycle as many times as we want!
Pumping Lemma – Formal Logic

If $A$ is a regular language then there is a number $p$ (the pumping length) where if $s$ is any string in $A$ of length at least $p$ then $s$ may be divided into three pieces, $s = xyz$ such that

- $|y| > 0$, and  // The loop is nonempty
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- If $A$ is a regular language then...
  - $\exists p\ (\forall s \in A\ |s| \geq p \rightarrow \exists x, y, z\ (s = xyz \land |y| > 0 \land |xy| \leq p \land (\forall i \in N\ xy^iz \in A))$

- Contrapositive: Negate both sides and swap them

- If $\forall p(\exists s \in A\ |s| \geq p \land \forall x, y, z\ ((s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow (\exists i \in N\ xy^iz \notin A))$
  - ...then $A$ is a nonregular language
  - If we can show that all values of $p$ are not pumping lengths for $A$ then we have shown that $A$ is nonregular
Pumping Lemma – Strategy

In proofs of nonregularity of language $A$ using the pumping lemma our goal is to show

$$\forall p (\exists s \in A \mid s \geq p \land \forall x, y, z \ (s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow (\exists i \in \mathbb{N} \ xy^i z \notin A))$$

- For any $p$

- There is a string $s$ such that

- For any viable split of the string into $x$, $y$, and $z$

- We can choose some # of repetitions of $y$ to get a string not in the language
Pumping Lemma – Strategy

In proofs of nonregularity of language $A$ using the pumping lemma our goal is to show

$$\forall p (\exists s \in A |s| \geq p \land \forall x, y, z \left( (s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow (\exists i \in N xy^{i}z \notin A) \right))$$

- For any $p$
  - Consider arbitrary $p$

- There is a string $s$ such that
  - Choose a string $s$ in terms of $p$ (creative part)

- For any viable split of the string into $x$, $y$, and $z$
  - Define $x$, $y$, $z$ according to PL conditions $|y| > 0 \land |xy| \leq p$

- We can choose some # of repetitions of $y$ to get a string not in the language
  - Choose $i$ such that $xy^{i}z$ is not in the language (other creative part)
Pumping Lemma – Example

- Consider the language \( PAL = \{w \in \{0,1\}^* | w = w^R\} \), i.e. the set of all palindromes over \( \{0,1\} \)
- Show that \( PAL \) is nonregular using the pumping lemma

- WTS

\[
\forall p (\exists s \in PAL \ |s| \geq p \land \forall x, y, z \ (s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow (\exists i \in N \ xy^i z \notin PAL))
\]

Consider arbitrary pumping length \( p \). WTS there is a valid string in \( PAL \) that can’t be pumped.

Which string should we choose?

A. 111000111
B. \( 10^p 1 \)
C. \( 0^p 10^p \)
D. \( 0^p 1^p \)
Pumping Lemma – Example

- Consider the language $PAL = \{w \in \{0,1\}^* | w = w^R\}$, i.e. the set of all palindromes over \{0,1\}

- WTS

  $\forall p (\exists s \in PAL \; |s| \geq p \land \forall x, y, z \left( (s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow (\exists i \in N \; x y^i z \notin PAL) \right))$

Consider arbitrary pumping length $p$. WTS there is a valid string in $PAL$ that can’t be pumped.

Consider string $s = 0^p 1^p \in PAL$, where $|s| > p$, as desired.

Let $s = xyz$ where $x = 0^k$, $y = 0^j$, $z = 0^l 1^p$ such that $j > 0$, and $k + j + l = p$.

WTS there is a value $i$ such that $xy^i z \notin PAL$

Consider $i = 0$. Then $xy^i z = xz = 0^k 0^j 1^p$. Since $j > 0$ then $k + l < k + j + l$.

Then $k + l < p$, so $0^k 0^j 1^p$ has an unequal number of leading and ending 0s, and therefore is not palindromic.

Therefore, $xy^0 z \notin PAL$ and $p$ is not a pumping length for $PAL$. Thus $PAL$ has no pumping length and is nonregular.
Pushdown Automata (PDA)

- What is the source of memory of an NFA?
  - The state it is in
  - That’s it
- Now add stack
  - we now have two sources of memory
PDA Formal Description

**Definition 2.13**

A *pushdown automaton* is a 6-tuple \((Q, \Sigma, \Gamma, \delta, q_0, F)\), where \(Q, \Sigma, \Gamma, \text{ and } F\) are all finite sets, and

1. \(Q\) is the set of states,
2. \(\Sigma\) is the input alphabet,
3. \(\Gamma\) is the stack alphabet,
4. \(\delta: Q \times \Sigma_{\varepsilon} \times \Gamma_{\varepsilon} \rightarrow \mathcal{P}(Q \times \Gamma_{\varepsilon})\) is the transition function,
5. \(q_0 \in Q\) is the start state, and
6. \(F \subseteq Q\) is the set of accept states.
**Compare And Contrast**

**Definition 2.13**

A **pushdown automaton** is a 6-tuple \((Q, \Sigma, \Gamma, \delta, q_0, F)\), where \(Q\), \(\Sigma\), \(\Gamma\), and \(F\) are all finite sets, and

1. \(Q\) is the set of states,
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\[
\delta_{PDA}(\text{state, char, pop}) = \{(\text{new state, push}), \ldots\}
\]

**Definition 1.37**

A **nondeterministic finite automaton** is a 5-tuple \((Q, \Sigma, \delta, q_0, F)\), where

1. \(Q\) is a finite set of states,
2. \(\Sigma\) is a finite alphabet,
3. \(\delta: Q \times \Sigma \rightarrow \mathcal{P}(Q)\) is the transition function,
4. \(q_0 \in Q\) is the start state, and
5. \(F \subseteq Q\) is the set of accept states.

\[
\delta_{NFA}(\text{state, character}) = \{\text{new state, \ldots}\}
\]
What are the following?:

\[
\delta(q_1, 0, \varepsilon) = \{(q_1, 0), (q_2, \varepsilon)\}
\]

\[
\delta(q_2, 1, 1) = \{(q_2, \varepsilon)\}
\]

\[
\delta(q_2, 0, \varepsilon) = \emptyset
\]
Convert Languages to PDA

\{a_1b_1a_2b_2...a_nb_n \mid \text{count}(a_1a_2...a_n) = \text{count}(b_1b_2...b_n) \land b_1b_2...b_n \in L(0^*1^*)\}

- Need to keep track of 1s in even positions and make sure they match the number of 1s in odd positions
- All 1s in odd positions need to come after 0s
- Empty string is allowed

Sample machine
Convert Languages to PDA

\[ \{a_1b_1a_2b_2\ldots a_nb_n \mid \text{count}(a_1a_2\ldots a_n) = \text{count}(b_1b_2\ldots b_n) \land b_1b_2\ldots b_n \in L(0^*1^*)\} \]
Decipher PDA Language
Decipher PDA Language

\[ \{ a^i b^j c^k \mid i = j \lor i = k \} \]